Completely reducible sets

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Abstract

We study the family of rational sets of words, called completely reducible and which are such that the syntactic representation of their characteristic series is completely reducible. This family contains, by a result of Reutenauer, the submonoids generated by bifix codes and, by a result of Berstel and Reutenauer, the cyclic sets. We study the closure properties of this family. We prove a result on linear representations of monoids which gives a generalization of the result concerning the complete reducibility of the submonoid generated by a bifix code to sets called birecurrent. We also give a new proof of the result concerning cyclic sets.

Contents

1 Introduction

The notion of syntactic algebra of a formal series has been introduced by Reutenauer in [?]. It is a natural generalization of the notion of syntactic monoid of a set of words. This algebra has a natural linear representation called the syntactic representation of the series, in the same way as the syntactic monoid has a natural representation by mappings from a set into itself corresponding to the minimal automaton of the set.

In the same way that one uses properties of the syntactic monoid of a set to define or characterize important classes of sets, it is natural to use the syntactic algebra to do the same. One of the most elementary property of a linear representation is its irreducibility or, more interestingly its complete reducibility. The syntactic representation and the syntactic algebra of a set of words is those of its characteristic series. A set of words is called completely reducible if its syntactic representation is completely reducible. This is equivalent to the semisimplicity of its syntactic algebra.

A remarkable property, also proved by Reutenauer in [?] is that, when the field is of characteristic zero, the submonoid generated by a rational bifix code is completely reducible. This can be considered as a generalization of Maschke's theorem and is one of the arguments showing the strong connexion between bifix codes and groups. Later, Berstel and Reutenauer proved in [?] that the sets of words called cyclic, are also completely reducible. The proofs given for both cases do not have much in common. The proof given in [?] for the first result consists in proving that the radical of the syntactic algebra of the set is zero. Another proof, given in [?] and in [?], shows directly that the syntactic representation of the set is completely reducible. This is also the proof presented in [?]. The proof of the other result on cyclic languages uses a decomposition of the characteristic series as a $\mathbb{Z}$-linear combination of series also called cyclic.

In this paper, we investigate further the family of completely reducible sets. We study the closure properties of this family (Theorem 2) and prove some necessary and some sufficient conditions to belong to the family. We characterize the completely reducible sets on a one letter alphabet (Theorem 3). Reworking the proof of the complete reducibility of submonoids generated by bifix codes, we prove a result on linear representation of semigroups which gives a sufficient condition for complete reducibility (Theorem 1). It is related with the theorem of Munn-Ponizovsky characterizing the finite 0-simple semigroups which have a semisimple algebra [?]. We use Theorem 1 to obtain a generalization of the complete reducibility of the submonoid generated by a bifix code to a class of sets called birecurrent (Theorem 4). They are defined by the property that the minimal automata of the set and of its reversal are strongly connected. Finally, we give a proof of the complete reducibility of cyclic sets which uses the notion

of external power of an automaton introduced by Béal in [?], the results on strongly cyclic sets proved in [?] and the family of series defined by traces used in the original proof of [?].

The problem of characterizing the completely reducible sets in terms of operations on sets of words remains open. It is solved one a one-letter alphabet and for the class of submonoids generated by rational maximal codes, since in this case the complete reducibility can only occur for a bifix code by the result of Reutenauer already mentionned. Such a characterization should take in account the characteristic of the field since the complete reducibility of the submonoid generated by a bifix code is only true when the characteristic of the field is zero (or more generally does not divide the order of the group of the bifix code).

The paper is organized as follows. In Section 2 we prove the result concerning completely reducible linear representation of monoids (Theorem 1). In Section 3 we define syntactic representations and recall some results concerning them. In Section 4, we prove some closure properties the family of completely reducible sets. We characterize this family on a one-letter alphabet (Theorem 3). In Section 5, we give a proof of the complete reducibility of birecurrent sets (Theorem 4). In Section 6 we give a new proof of the complete reducibility of cyclic sets (Theorem 5).

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2 Completely reducible monoids of matrices

In the first two parts of this section, we introduce basic notions concerning monoids and linear representations. For a more detailed exposition, see [?] or [?]. In the last part, we prove a result on completely reducible monoids of matrices (Theorem 1) which will be used in Section 5.

2.1 Monoids

A semigroup is a set with an associative operation. A monoid is a semigroup with an identity element denoted 1. If S is a semigroup, we denote by S^1 the monoid which is equal to S if S is a monoid and to $S \cup 1$ otherwise.

An element 0 of a monoid M is a *zero* if for all $m \in M$ $0m = m0 = 0$. If M contains a zero, it is unique.

The Green relations on a monoid M are defined as follows. For $m, n \in M$, one has

(i) $m\mathcal{R}n$ if $mM = nM$.

(ii) $m\mathcal{L}n$ if $Mm = Mn$.

(iii) $m\mathcal{H}n$ if $m\mathcal{R}n$ and $m\mathcal{L}n$.

It is classical that $\mathcal{R}$ and $\mathcal{L}$ commute. One denotes $\mathcal{D}$ the relation $\mathcal{R}\mathcal{L} = \mathcal{L}\mathcal{R}$. The $\mathcal{R}$-class of $m \in M$ is denoted $R(m)$ and similarly for $\mathcal{L}$, $\mathcal{H}$ and $\mathcal{D}$.

A $\mathcal{D}$-class D is *regular* if it contains an idempotent. In this case, there is an idempotent in each $\mathcal{R}$-class and in each $\mathcal{L}$-class of D.

For any $m, n \in M$ one has $mn \in R(m) \cap L(n)$ if and only if $R(n) \cap L(m)$ contains an idempotent (Clifford-Miller's lemma). As a consequence, for any m, m' in the same $\mathcal{H}$-class H, either $mm' \notin H$ or H is a group.

A *right ideal* (resp. a left ideal, resp. a two-sided ideal) of a monoid M is a nonempty subset I such that $IM = I$ (resp. $MI = I$, resp. $MIM = I$). A right ideal is *minimal* if it does not contain any other right ideal of M (and similarly for left and for two-sided ideals). In a finite monoid, there is a unique minimal two-sided ideal which is the union of minimal right ideals (resp. of minimal left ideals). When M contains a zero, a right ideal $I \neq 0$ is 0-minimal if the only right ideals contained in I are 0 and I itself (and similarly for left and two-sided ideals).

A semigroup S is called *simple* if the only two-sided ideal contained in S is S itself. A semigroup with zero is 0-simple if $S^2 \neq 0$ and 0 is the only proper two-sided ideal of S.

A *Rees matrix semigroup* $\mathcal{M}^0(G, I, J, P)$ is given by a finite group G, two finite sets I, J and a $J \times I$-matrix P with elements in $G \cup 0$ called the *sandwich matrix*. The semigroup is formed of 0 and of all triples $(i, g, j) \in I \times G \times J$ and the multiplication is given by $(i, g, j)(k, h, \ell) = (i, gP_{j,k}h, \ell)$. The semigroup is regular if and only if P has no null row or null column.

If P contains no zero entry, the set $\mathcal{M}\backslash 0$ is a semigroup denoted $\mathcal{M}(G, I, J, P)$.

By Rees theorem, a finite semigroup is 0-simple if and only if it is isomorphic to a regular Rees matrix semigroup.

2.2 Linear representations of monoids

Let V be a vector space over a field K and let M be a submonoid of the monoid $\mathrm{End}(V)$ of linear functions from V into itself. A subspace V' of V is *invariant* by M if $V'm \subset V'$ for any $m \in M$. The monoid M is called *reducible* if there is a subspace V' of V which is invariant by M and such that $V' \neq 0, V' \neq V$. Otherwise, M is called *irreducible*.

The monoid M is *completely reducible* if any invariant subspace has an invariant supplement. If V has finite dimension, a completely reducible submonoid of $\mathrm{End}(V)$ has the following form. There exists a decomposition of V into a direct sum of invariant subspaces $V_1, V_2, \ldots, V_k$,

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_k$$

such that the restrictions of the elements of M to each of the V_i's form an irreducible submonoid of $\mathrm{End}(V_i)$. Any invariant subspace of V is a sum of one or more of the V_i. Conversely, if V is of this form, then M is completely reducible.

In a basis of V composed of bases of the subspaces V_i, the matrix of an element m in M has a diagonal form by blocks,

$$m = \begin{bmatrix} m_1 & & & 0 \\ & m_2 & & \\ & & \ddots & \\ 0 & & & m_k \end{bmatrix}.$$

The subspaces V_i are called the *irreducible components* of V under M. The restrictions of M to the subspaces V_i are called the *irreducible constituents* of M.

Let M be a monoid and let V be a vector space over a field K. A *linear representation* of M over V is a morphism φ from M into $\mathrm{End}(V)$. A subspace W of V is invariant under φ if it is invariant under $\varphi(M)$. The representation is completely reducible if the monoid $\varphi(M)$ is completely reducible. The irreducible components and the irreducible constituents of φ are those of $\varphi(M)$.

An algebra is said to be *simple* if it has no other two-sided ideals than 0 and itself. It is said to be *semisimple* if it is a finite direct product of simple algebras.

We will use some well-known properties of semisimple algebras (see [**?**] or [**?**] for example).

First a quotient of a semisimple algebra is semisimple and a direct product of semisimple algebras is semisimple. Note that a subalgebra of a semisimple algebra is not semisimple in general since any finite dimensional algebra over a field K is a subalgebra of the algebra $K^{n \times n}$ of $n \times n$-matrices, which is simple.

Next, by Wedderburn's theorem, every semisimple algebra contains an identity element. Finally a nilpotent ideal of a semisimple algebra is zero.

A representation of an algebra over a vector space V is a morphism φ from $\mathfrak{A}$ into the algebra $\mathrm{End}(V)$. It is *faithful* if φ is injective. The representation is reducible or completely reducible if $\varphi(\mathfrak{A})$ is reducible or completely reducible respectively.

As well known, the properties of an algebra of being simple or semisimple correspond to the properties of their representations to be irreducible or completely reducible respectively. More precisely, if an algebra has a faithful irreducible representation, then it is simple. If it has a faithful completely reducible representation, then it is semisimple. Conversely, any representation of a simple algebra is irreducible and every representation of a semisimple algebra is completely reducible (see [**?**] for example).

We recall that, by Maschke's theorem a linear representation of a finite group on a field of characteristic zero is completely reducible.

2.3 A sufficient condition for complete reducibility

In the statement below, we consider a monoid M of $n \times n$-matrices acting on the right on the space V of row n-vectors. This is of course equivalent to considering a submonoid of $\mathrm{End}(V)$ by taking a basis of V.

Theorem 1 *For $n \geq 1$, let V (resp. W) be the vector space of row (resp. column) n-vectors on a field K. Let M be a monoid of $n \times n$-matrices on K. Assume that there is an idempotent e in M such that the following conditions are satisfied.*

(i) *The linear representation of eMe by restriction to Ve is completely reducible.*

(ii) *The space V is generated by the set $VeM = \{vem \mid v \in V, m \in M\}$.*

(iii) *The space W is generated by the set $MeW = \{mew \mid w \in W, m \in M\}$.*

Then M is completely reducible. Moreover, the number of irreducible components of V under M and of Ve under eMe are the same.

Proof Set $T = Ve$. Let V' be a subspace of V which is invariant by M and let $T' = V'e$. By condition (i), there is a supplement T'' of T' which is invariant by eMe. We define the subspace

$$V'' = \{v \in V \mid vMe \subset T''\}$$

with $vMe = \{vme \mid m \in M\}$. We verify that V'' is an invariant supplement of V'.

First, we have for any $v \in V''$ and $m \in M$, $vmMe \subset vMe \subset T''$, which implies $vm \in V''$. Thus V'' is invariant.

Next, $V = V' + V''$. Indeed, consider $v \in V$. By condition (ii), v is a linear combination of vectors of the form $v_m em$ for some $v_m \in V$ and $m \in M$. Since $v_m e$ is in T for each $m \in M$, there are $v'_m \in T'$ and $v''_m \in T''$ such that $v_m e = v'_m e + v''_m e$. Then for some $\alpha_m \in K$,

$$v = \sum_{m \in M} \alpha_m v_m em = \sum_{m \in M} \alpha_m v'_m em + \sum_{m \in M} \alpha_m v''_m em.$$

Since $v'_m \in T' \subset V'$ and since V' is invariant, the first term is in V'. Since T'' is invariant by eMe, $v''_m emMe \subset v''_m eMe \subset T''$, the second one is in V''. Thus v is in $V' + V''$.

Finally, $V' \cap V'' = 0$. Let indeed $v \in V' \cap V''$. For each $m \in M$, we have $vme \in T' \cap T''$ and thus $vme = 0$. By condition (iii), for any $w \in W$, we have $w = \sum_{m \in M} \alpha_m mew_m$ for some $w_m \in W$ and $\alpha_m \in K$. Then

$$vw = v \sum_{m \in M} \alpha_m mew_m = \sum_{m \in M} \alpha_m vmew_m = 0$$

This implies that $v = 0$, which completes the proof that $V' \cap V'' = 0$.

To prove the last assertion, consider a decomposition $T = T_1 \oplus T_2 \oplus \ldots \oplus T_k$ of T into a direct sum of irreducible invariant subspaces. The invariant subspaces of T are then the sums of some of the T_i. For $1 \leq i \leq k$, let $V_i = \{v \in V \mid vMe \subset T_i\}$. Then the above proof shows that $V = V_1 \oplus V_2 \oplus \ldots \oplus V_k$ is a decomposition of V into a direct sum of irreducible invariant subspaces. Indeed, the sum is

direct because for each i, $V_1 + \ldots + V_{i-1} + V_{i+1} + \ldots + V_k$ is a supplement of V_i. And each V_i is irreducible because if V_i' is an invariant subspace of V_i, then $T_i' = V_i' \cap T$ is an invariant subspace of T_i, wich implies $T_i' = 0$ or $T_i' = T_i$. If $T_i' = T_i$, then V_i' and V_i have a common supplement V_i'' and thus the dimensions of V_i and V_i' are equal. Since $V_i' \subset V_i$, we conclude that $V_i = V_i'$. Similarly, if $T_i' = 0$, V_i' has a supplement equal to V and thus $V_i' = 0$. ■

We relate the previous result with the classical theory of linear representations of semigroups (see [**?**]). The algebra of a semigroup S on a field K is the set $K[S]$ of linear combinations of elements of S with the product extending the product of S. Let S be a semigroup with zero element z. The *contracted algebra* of S over a field K is the algebra $K_0[S]$ formed of linear combinations of elements of S with z identified with 0.

An important particular case of Theorem 1 occurs when M is the union of the identity with a finite 0-simple semigroup isomorphic to a regular Rees matrix semigroup $\mathcal{M}^0(G, I, J, P)$.

The *Munn algebra* corresponding to M is the algebra $\mathcal{M}(K[G], I, J, P)$ of $I \times J$ matrices with coefficients in $K[G]$ with the multiplication

$$u \circ v = uPv$$

The contracted algebra of $\mathcal{M}^0(G, I, J, P)$ on K is isomorphic with the Munn algebra $\mathcal{M}(K[G], I, J, P)$.

By a result of Munn-Ponizovsky (see [**?**], Theorem 5.19) the Munn algebra $\mathcal{M}(K[G], I, J, P)$ is semisimple if and only if $K[G]$ is semisimple and P is invertible.

Assume that M is a monoid which is the union of the identity and a 0-simple semigroup S isomorphic to $\mathcal{M}^0(G, I, J, P)$. Let $\varphi : M \to K^{n \times n}$ be a linear representation of M by $n \times n$-matrices on K. It is not true in general, even if φ is injective, that the algebra generated by $\varphi(S)$ is isomorphic with the Munn algebra $\mathcal{M}(K[G], I, J, P)$. In particular, the fact that φ is completely reducible does not imply that $\mathcal{M}(K[G], I, J, P)$ is semisimple (see Example 1). However, the converse is true since the algebra generated by $\varphi(S)$ is a quotient of $\mathcal{M}(K[G], I, J, P)$. In this case the conditions of Theorem 1 are also necessary, that is, if a monoid of matrices of the form $M = S^1$, with S a finite 0-simple semigroup (0 being the null matrix), is completely reducible then conditions (i), (ii) and (iii) in Theorem 1 are satisfied.

Indeed, let e be any nonzero idempotent in S. Since $S \setminus 0$ is a $\mathcal{D}$-class, the monoid eMe is the union of a group isomorphic to G and 0. Since K is of characteristic zero, eMe is completely reducible. Thus condition (i) is satisfied. Since S is completely reducible, the algebra generated by S is semisimple. Since any semisimple algebra contains an identity element, the algebra generated by S contains the identity matrix. Let $\sum_{s \in S} \alpha_s s$ be the identity. For any $v \in V$, we have $v = \sum_{s \in S} \alpha_s v s$. Since e, s are in the same $\mathcal{D}$-class, the are $m_s, n_s \in M$ such that $s = n_s e m_s$. Then $v = \sum_{s \in S} \alpha_s (v n_s) e m_s$ is in the subspace of V generated by VeM. This shows that (ii) holds. A symmetric proof shows that (iii) also holds.

Example 1 Let M be the monoid generated by the two matrices with integer coefficients

$$\alpha = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \beta = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

We consider the elements of M as $Q \times Q$-matrices with $Q = \{1, 2, 3, 4\}$ with coefficients in the field K of rational numbers. Each element of M acts on the right on the space $V = K^Q$ of row Q-vectors. We identify each index $1, 2, 3, 4$ with its characteristic row vector. Thus, for example, $2 + 3 - 4$ is the same as $[0\ 1\ 1\ \ -1]$.

The monoid M is formed of the identity and a regular $\mathcal{D}$-class of elements of rank 3 which is its minimal ideal. It is represented in Figure 1.

	$1, 2, 3$	$1, 4, 2 + 3 - 4$
$2 - 4$	* α	* $\alpha\beta$
$3 - 4$	* $\beta\alpha$	* β

Figure 1: The minimal ideal of M.

We indicate above each $\mathcal{L}$-class a basis of the common image of its elements (recall that, in a monoid of matrices, two $\mathcal{L}$-equivalent elements have the same image). Thus the set $\{1, 4, 2+3-4\}$ placed above the second column is a basis of the image of the elements in $L(\beta)$ (these vectors are actually the distinct rows of the matrix β). Similarly, we indicate on the left of each $\mathcal{R}$-class a basis of the common kernel of its elements (recall that in a monoid of matrices, two $\mathcal{R}$-equivalent elements have the same kernel). The star placed in a $\mathcal{H}$-class indicates that it is a group. Here, all four $\mathcal{H}$-classes are groups.

The element $e = \alpha^3$ is an idempotent. The set eMe, which is the $\mathcal{H}$-class of α, is a group G. It is faithfully represented by its action on the image of e. This action realizes a permutation of the three vectors $1, 2, 3$. Indeed, the action of α, β maps bijectively the sets of vectors $1, 2, 3$ and $1, 4, 2 + 3 - 4$ respectively as indicated in Figure 2. Since α realizes the permutation (123) and $\alpha\beta\alpha$ the transposition (12), and since $\alpha, \alpha\beta\alpha \in H(\alpha)$ the group G is the symmetric group on $\{1, 2, 3\}$.

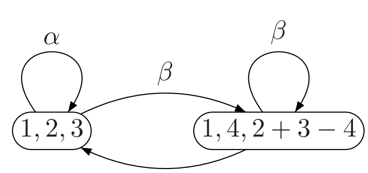

Figure 2: The action of α and β.

The conditions of Theorem 1 are satisfied with $e = \alpha^3$. Condition (i) is satisfied because eMe is a finite group and K is of characteristic 0. Condition (ii) is satisfied because the union of the images of α and β generate V and condition (iii) is satisfied because the intersection of the kernels of α and β is zero. Thus M is completely reducible by Theorem 1. Let us describe in more detail the structure of the monoid M.

The linear representation of the symmetric group associated to its representation by permutations on the set $\{1, 2, 3\}$ has only two invariant subspaces. The first one is the one-dimensional space generated by the vector $1 + 2 + 3$. The second one is the two-dimensional space generated by the vectors $1 - 2$ and $2 - 3$. The corresponding invariant subspaces of V are the subspace generated by $1+2+3$ and the subspace formed of the vectors with zero sum of coordinates. In the basis

$$1 + 2 + 3, \quad 1 - 2, \quad 2 - 3, \quad 1 - 4$$

the matrices α, β take the form

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 2 & 1 & -1 \\ 0 & 2 & 1 & -2 \end{bmatrix}$$

Thus the monoid M is a direct sum of two irreducible monoids of matrices of dimensions 1 and 3.

Note that the algebra of the monoid M is not completely reducible. Indeed, the monoid M is the union of 1 and a simple semigroup which is isomorphic with the regular Rees matrix semigroup $\mathcal{M} = (G, I, J, P)$ with $I = J = \{1, 2\}$, G the symmetric group on $\{1, 2, 3\}$ and

$$P = \begin{bmatrix} (1) & (1) \\ (1) & (23) \end{bmatrix}$$

The isomorphism uses the four idempotents $e_{11} = \alpha^3$, $e_{12} = (\alpha\beta)^2$, $e_{21} = (\beta\alpha)^2$ and $e_{22} = \beta^3$. in such a way that $P_{ji} = e_{j1}e_{1i}$. It is defined for $m \in H(e_{ij})$ by $m \mapsto (e_{11}me_{11}, i, j)$. Thus

$$\alpha \mapsto \begin{bmatrix} (123) & 0 \\ 0 & 0 \end{bmatrix}, \quad \beta \mapsto \begin{bmatrix} 0 & 0 \\ 0 & (12) \end{bmatrix},$$

The sandwich matrix P is not invertible. Indeed, the difference of its two rows is the vector $[0 \quad 1-(23)]$. But $1-(23)$ is a divisor of zero since $(1-(23))(1+(23)) = 0$. Setting $x = 1 + (23)$ and $v = [x \ -x]$, we obtain $vP = 0$. Thus the Munn algebra $(K[G], I, J, P)$ is not semisimple.

3 Automata and syntactic representations

In this section, we introduce the basic terminology concerning automata and syntactic representations. For a more detailed exposition, see [**?**] or [**?**].

3.1 Words and formal series

Let A be a finite set called an alphabet. We denote by A^* the set of words on A and by A^+ the set of nonempty words.

For a word $w = a_1a_2 \cdots a_n$ with $a_i \in A$, we denote by $\tilde{w}$ the *reversal* of w. By definition $\tilde{w} = a_n \cdots a_2a_1$. By convention $\tilde{1} = 1$. For a set X of words, the reversal of X is the set $\tilde{X} = \{\tilde{x} \mid x \in X\}$.

For two words $x, y \in A^*$, we define $x^{-1}y = z$ if $y = xz$ and $x^{-1}y = \emptyset$ otherwise. Symmetrically $xy^{-1} = z$ if $x = zy$ and $xy^{-1} = \emptyset$ otherwise. The notation is extended to sets by linearity. Thus for example $x^{-1}Y = \{z \in A^* \mid xz \in Y\}$.

A word v is a factor of a word x if $x = uvw$ for some words u, w. For a set X of words, we denote by $F(X)$ the set of factors of the words of X.

Let K be a field. A *formal series* S on the alphabet A with coefficients in K is a map $S : A^* \to K$. For $w \in A^*$, we denote by (S, w) the value of S on w. The value $(S, 1)$ is called the *constant term* of S.

The sum of S and T is the series defined by $(S + T, w) = (S, w) + (T, w)$. Likewise, for $\alpha \in K$, the series αS is defined by $(\alpha S, w) = \alpha(S, w)$. In this way the set of formal series becomes a vector space.

We denote by $K\langle A\rangle$ the free algebra on A. Its elements, called *polynomials*, are formal series such that all but a finite number of the coefficients are 0. When the alphabet has one letter a, we use the traditional notation $K[a]$ rather than $K\langle a\rangle$.

For a set $X \subset A^*$, we denote by $\underline{X}$ the characteristic series of X, which is defined by $(\underline{X}, x) = 1$ if $x \in X$ and 0 otherwise.

Let $n \geq 1$ be an integer. Let λ be a row n-vector, let μ be a morphism from A^* into the monoid of $n \times n$-matrices and let γ be a column n-vector, all with coefficients in K. The triple (λ, μ, γ) is said to be a *linear representation* of a series S if for any word w,

$$(S, w) = \lambda\mu(w)\gamma.$$

We also say that (λ, μ, γ) *recognizes* S. The vector λ is called the *initial vector* and γ the *terminal vector*. The series S is said to be *rational* if it has a linear representation.

We say that a morphism ψ from the free algebra $K\langle A\rangle$ into an algebra $\mathfrak{A}$ *recognizes* a series S if there is a linear map $\pi : \mathfrak{A} \to K$ such that $(S, w) = \pi(\psi(w))$ for all $w \in A^*$.

Let S be a rational series and let (λ, μ, γ) be a linear representation of S. Then μ extends to a morphism from $K\langle A\rangle$ into the algebra $K^{n\times n}$ of $n \times n$-matrices with coefficients in K. This morphism recognizes S since the linear map $\pi : K^{n\times n} \to K$ defined by $\pi(m) = \lambda m\gamma$ satisfies $(S, w) = \pi(\mu(w))$ for any $w \in A^*$.

Conversely, one can recover a linear representation of a rational series S from a morphism ψ into a finite dimensional algebra $\mathfrak{A}$ recognizing S. Indeed, let us choose a basis of $\mathfrak{A}$. Then the map $x \to x\psi(w)$ from $\mathfrak{A}$ into itself is linear. It is represented by an $n \times n$-matrix $\mu(w)$. Let λ be the n-vector representing $\psi(1)$

in this basis. Let γ be a column n-vector γ such that $\pi(x) = x\gamma^t$ for any x in $\mathfrak{A}$. Then $\lambda\mu(w)\gamma = (S, w)$ for all $w \in A^*$, showing that (λ, μ, γ) is a linear representation of S.

This shows the following useful equivalent definition of rational series.

Proposition 1 *A series is rational if and only if it can be recognized by a morphism into a finite dimensional algebra.*

Example 2 Let $X = (a^2)^*$ and let S be the series $S = \underline{X}$. Then S is recognized by the linear representation (λ, μ, γ) whith $\lambda = [1\ 0]$, $\gamma = [1\ 0]^t$ and μ defined by $\mu(a) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. It is also recognized by the morphism μ with the linear map from $K^{2\times 2}$ in K defined by $\pi(x) = x_{1,1}$.

3.2 Automata

An automaton $\mathcal{A} = (Q, I, T)$ on the alphabet A is composed with a finite set Q of states, a set $I \subset Q$ of initial states, a set $T \subset Q$ ot terminal states and a set $E \subset Q \times A \times Q$ of edges. If (p, a, q) is an edge we say that it starts at p, it ends at q and that a is its label. Two edges (p, a, q) and (p', a', q') are consecutive if $q = p'$. A *path* from p to q in the automaton is a sequence $c : p \xrightarrow{a_1} q_1 \to \cdots \to q_{n-1} \xrightarrow{a_n} q$ of consecutive edges. The word $w = a_1 \cdots a_n$ is its label. We denote such a path $c : p \xrightarrow{w} q$. A word w is recognized by the automaton $\mathcal{A}$ if there is a path labeled w from a state in I to a state in T. Two automata are called *equivalent* if they recognize the same set of words.

A set of words X is *rational* if it is the set of words recognized by an automaton.

An automaton $\mathcal{A} = (Q, I, T)$ is *deterministic* if $\mathrm{Card}(I) \leq 1$ and for each $p \in Q$ and each $a \in A$ there is at most one edge starting at p and labeled a. In this case, there is a partial map from $Q \times A$ to Q denoted $(q, a) \mapsto q \cdot a$. The maps $(q, a) \mapsto q \cdot a$ are called the *transitions* of the automaton. This map is extended to a partial map from $Q \times A^*$ to Q also denoted $(q, w) \mapsto q \cdot w$ for $q \in Q$ and $w \in A^*$.

A deterministic automaton with a unique initial state i will be denoted $\mathcal{A} = (Q, i, T)$. This notation implies in particular that Q is not empty. Unless otherwise stated, all automata considered in this paper are deterministic.

Let $\mathcal{A} = (Q, i, T)$ be a deterministic automaton. A state $q \in Q$ is *accessible* if there is a word u such that $i \cdot u = q$ and *coaccessible* if there is a word v such that $q \cdot v \in T$. The automaton is *trim* if every state is accessible and coacessible. For any automaton, $\mathcal{A} = (Q, i, T)$ the automaton obtained by suppressing all states which are not accessible is called the *accessible part* of $\mathcal{A}$. The automaton obtained by suppressing all states which are not accessible and coacessible is the *trim part* of $\mathcal{A}$. Both automata are equivalent to $\mathcal{A}$.

Any automaton can be converted into a deterministic equivalent one. Indeed, let $\mathcal{A} = (Q, I, T)$ be an automaton with a set E of edges. Let $\mathcal{B}$ be the automaton having as states the nonempty subsets of Q. Its transitions are defined, for

$U \subset Q$ and $a \in A$, by $U \cdot a = \{q \in Q \mid (u, a, q) \in E$ for some $u \in U\}$ if this set is nonempty. Using the set I as initial state and the family $\mathcal{T} = \{U \subset Q \mid U \cap T \neq \emptyset\}$ as set of terminal states, one obtains a deterministic automaton equivalent to $\mathcal{A}$. The above construction is called the *subset construction*. The automaton obtained by taking the accessible part of the result is said to be obtained by the *accessible subset contruction*.

EXAMPLE 3 Let $\mathcal{A}$ be the automaton represented in Figure 3 on the left. The initial state is 1 which is the unique terminal state. An initial state is indicated by an incoming edge and a terminal state by an outgoing one. The automaton is not deterministic because there are two edges labeled a going out of state 1. The result of the accessible subset construction is indicated in Figure 3 on the right.

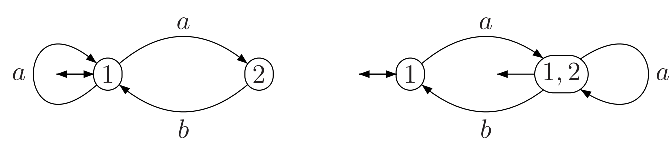

Figure 3: A nondeterministic automaton and the result of the accessible set construction.

We will occasionally consider a more general notion of automaton, called a *weighted automaton*. Let K be a field. A weighted automaton $\mathcal{A} = (Q, I, T)$ with weights in K is given by two maps $I, T : Q \to K$ and a map $E : Q \times a \times Q \to K$. If $E(p, a, q) = k \neq 0$, we say that (p, a, q) is an edge with label a and weight $E(p, a, q)$ and we write $p \overset{ka}{\to} q$.

For a given automaton $\mathcal{A}$, any word $w \in A^*$ defines a partial map $\varphi_{\mathcal{A}}(w) : q \mapsto q \cdot w$ from Q into itself. The monoid $\varphi_{\mathcal{A}}(A^*)$ is the *transition monoid* of the automaton.

The *minimal automaton* of a set $X \subset A^*$ is the automaton with set of states the nonempty sets $u^{-1}X$ for $u \in A^*$ and the transitions $(u^{-1}X, a) \mapsto (ua)^{-1}X$. The state $1^{-1}X = X$ is the initial state and the set of final states is the set of $u^{-1}X$ such that $u \in X$ or, equivalently, $1 \in u^{-1}X$. The minimal automaton of X is trim and recognizes X.

The minimal automaton of the empty set has an empty set of states. In all other cases, the minimal automaton has a nonempty set of states and a unique initial state.

Let $X \subset A^*$. The set of *contexts* of a word w is the set $C(w) = \{(u, v) \in A^* \mid uwv \in X\}$. The *syntactic congruence* is the equivalence defined by $w \equiv w'$ if $C(w) = C(w')$. The *syntactic monoid* of X is the quotient of A^* by the syntactic congruence. One sometimes needs to use rather the *synstactic semigroup* of a set $X \subset A^+$ which is the quotient of A^+ by the syntactic congruence.

The syntactic monoid of a set X is isomorphic with the transition monoid of the minimal automaton of X.

Let $\mathcal{A} = (Q, i, T)$ be a deterministic automaton and let K be a field. The *linear representation associated* with $\mathcal{A}$ is the morphism from A^* into the monoid

of $Q \times Q$-matrices with coefficients in K defined for $a \in A$ by

$$\mu(a)_{p,q} = \begin{cases} 1 & \text{if } p \cdot a = q \\ 0 & \text{otherwise} \end{cases}$$

If $\mathcal{A}$ is a weighted automaton, we set $\mu(a)_{p,q} = \sum_{p \xrightarrow{ka} q} k$ for each $a \in A$ and $p, q \in Q$. The *trace* of a word w with respect to the automaton $\mathcal{A}$ is the trace of the matrix $\mu(w)$.

The characteristic series of a rational set of words is a rational series. Indeed, let $\mathcal{A} = (Q, i, T)$ be an automaton recognizing X. Let λ be the characteristic row Q-vector of i, let μ be the linear representation associated with $\mathcal{A}$ and let γ be the characteristic column Q-vector of the set T. Then (λ, μ, γ) is a linear representation of $\underline{X}$.

A deterministic automaton $\mathcal{A} = (Q, i, T)$ is *strongly connected* if for any $p, q \in Q$ there is a word w such that $p \cdot w = q$.

The following statement is well known (see for example [**?**] Proposition 8.2.5 or [**?**] Proposition 1.12.9).

Proposition 2 *Let $\mathcal{A} = (Q, i, T)$ be a deterministic strongly connected automaton. The transition monoid M of $\mathcal{A}$ has a unique 0-minimal or minimal two-sided ideal D according to the case where M has a zero or not, formed of the elements of nonzero minimal rank. It is a regular $\mathcal{D}$-class. For any $m \in D$, either $m^2 = 0$ or the $\mathcal{H}$-class of m is a group.*

Proof We prove the statement when M has a zero which is the empty map. The other case is similar. Let r be the minimal nonzero rank of the elements of M and let D be the set of elements of rank r.

The set $D \cup 0$ is clearly a two-sided ideal. We first remark that for any $m, n \in D$, there is an $u \in M$ such that $mun \neq 0$. Indeed, let $p, q, r, s \in Q$ be such that $pm = q$ and $rn = s$. Since $\mathcal{A}$ is strongly connected there is an element $u \in M$ such that $qu = r$. Then $pmun = s$ and thus $mun \neq 0$.

Let us first show that for each $m \in D$, the right ideal mM is 0-minimal. Let $u \in M$ be such that $mu \neq 0$. By the above remark, there exists $v \in M$ such that $muvm \neq 0$. Let $I = Qm$ be the image of m and let $z = uvm$. Since $z \in Mm$, we have $Iz \subset I$. And since $z \neq 0$, $Iz \subset I$ implies $Iz = I$ by minimality. Thus there is an integer $k \geq 1$ such that z^k is the identity on I. Set $e = z^k$ and $w = vmz^{k-1}$. Since e is the identity on Qm, one has $me = m$. Thus $m = muw$ showing that $mu \in R(m)$.

Let $m, n \in D$. By the above remark there is $u \in M$ such that $mun \neq 0$. Since mM is a 0-minimal right ideal, there exists $v \in M$ such that $munv = m$. Thus $mun \in R(m)$. The proof that $mun \in L(n)$ is symmetrical. This shows that D is a $\mathcal{D}$-class. Since two elements in the same $\mathcal{D}$-class generate the same two-sided ideal, D is a 0-minimal two-sided ideal.

Set $n' = un$. Then $mn' \in R(m) \cap L(n')$ implies by Clifford-Miller's lemma that $R(n') \cap L(m)$ contains an idempotent. Thus D is a regular $\mathcal{D}$-class.

For any $m \in D$ either $m^2 \in H(m)$ and $H(m)$ is a group by the Clifford-Miller's lemma, or $m^2 \notin H(m)$. In this case, since mM is a 0-minimal right ideal, we cannot have $m^2 \neq 0$. Thus $m = 0$. ∎

3.3 Syntactic representations

Let S be a formal series. For $u \in A^*$, we denote by $S \cdot u$ the series defined by $(S \cdot u, v) = (S, uv)$. The following formulas hold

$$S \cdot 1 = S, \quad (S \cdot u) \cdot v = S \cdot uv.$$

The *syntactic space* of S, denoted V_S, is the vector space generated by the series $S \cdot u$ for $u \in A^*$. The *syntactic representation* of S is the morphism $\psi_S : K\langle A\rangle \to \mathrm{End}(V_S)$ defined for $x \in V_S$ and $u \in A^*$ by

$$x\psi_S(u) = x \cdot u$$

The *syntactic algebra* of S, denoted $\mathfrak{A}_S$, is the image of the free algebra $K\langle A\rangle$ by the syntactic representation. The syntactic algebra of S can also be defined directly as follows. Denote by $p \mapsto (S, p)$ the extension of S to the free algebra on A. Then $\mathfrak{A}_S$ is the quotient of the free algebra by the equivalence

$$p \equiv 0 \ \Leftrightarrow \ (S, upv) = 0 \text{ for all } u, v \in A^*. \tag{1}$$

The morphism ψ_S recognizes S since the map π from $\mathfrak{A}_S$ into K defined by $\pi(\psi_S(w)) = (S, w)$ is well-defined and linear.

The syntactic algebra of a series S satisfies the following universal property (see [**?**], Exercise 2.1.4).

Proposition 3 *If $\psi : K\langle A\rangle \to \mathfrak{A}$ is a surjective morphism recognizing S, there exists a morphism ρ from $\mathfrak{A}$ onto the syntactic algebra of S such that $\psi_S = \rho \circ \psi$.*

Proof Let $\pi : \mathfrak{A} \to K$ be the linear map such that $\pi(\psi(w)) = (S, w)$ for any $w \in A^*$. We have to prove that if $p \in K\langle A\rangle$ is such that $\psi(p) = 0$, then $\psi_S(p) = 0$. But if $\psi(p) = 0$, then for any $u, v \in A^*$, $(S, upv) = \pi(\psi(upv)) = \pi(\psi(u)\psi(p)\psi(v))) = 0$ and thus $\psi_S(p) = 0$. ∎

Thus, in view of Proposition 1, a series is rational if and only if its syntactic algebra is finite dimensional.

A linear representation (λ, μ, γ) of a series S is said to be *minimal* if the dimension of the matrices $\mu(w)$ is equal to the dimension of V_S. In this case, for any word w, $\mu(w)$ is the matrix representing $\psi_S(w)$ in some basis.

Note that in this case K^n is generated by the vectors $\lambda\mu(w)$ for $w \in A^*$. Symmetrically the space W of column n-vectors is generated by the $\mu(w)\gamma$ for $w \in A^*$.

Example 4 Let $A = \{a\}$ and $S = \underline{a^+}$. Then $\{S, S \cdot a\}$ is a basis of V_S and S is recognized by the linear representation (λ, μ, γ) with $\lambda = [1\ 0]$, $\gamma = [0\ 1]^t$ and

$$\mu(a) = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

This representation is minimal.

The following is Proposition 14.7.1 in [**?**].

Proposition 4 *Let X be a subset of A^* and let $S = \underline{X}$. Let φ be the canonical morphism from A^* onto the syntactic monoid M of X. Then for all $u, v \in A^*$,*

$$\varphi(u) = \varphi(v) \Leftrightarrow \psi_S(u) = \psi_S(v)\,.$$

In particular the monoid $\psi_S(A^*)$ is isomorphic to M.

4 Completely reducible sets

In this section we define the family of completly reducible sets. In the first part, we prove some closure properties of this family (Theorem 2). In the second part, we prove some necessary and some sufficient conditions for membership in the family and a characterization in the case of a one-letter alphabet.

4.1 Completely reducible series

A series is *completely reducible* if its syntactic representation is completely reducible. As we have seen in Section 2, this is equivalent to the semisimplicity of its syntactic algebra. Moreover, by Proposition 3, if a series S is recognized by a morphism onto a semisimple algebra, then S is completely reducible.

The following result was suggested to me by Christophe Reutenauer (personal communication).

Proposition 5 *Any linear combination of completely reducible series is completely reducible.*

We use the following property.

Lemma 1 *Let φ_1, φ_2 be two morphisms from A^* into* $\mathrm{End}(V_1)$ *and* $\mathrm{End}(V_2)$ *respectively. Set $V = V_1 \times V_2$. If φ_1, φ_2 are completely reducible, the morphism φ from A^* into* $\mathrm{End}(V)$ *defined by $\varphi(w)(v_1, v_2) = (\varphi_1(w)(v_1), \varphi_2(w)(v_2))$ is completely reducible.*

Proof Since V_1 and V_2 are direct sums of irreducible components W_i, the same holds for V and thus φ is completely reducible. ∎

Proof of Proposition 5. If S is completely reducible, then αS is clearly reducible for any $\alpha \in K$. Next let S_1, S_2 be completely reducible series. For $i = 1, 2$,

let $\mathfrak{A}_i = \mathfrak{A}_{S_i}$, $\psi_i = \psi_{S_i}$ and let $\pi_i : \mathfrak{A}_i \to K$ be the linear map defined by $\pi_i(\psi_i(w)) = (S_i, w)$. Consider the morphism $\psi : A^* \to \mathfrak{A}_1 \times \mathfrak{A}_2$ defined by $\psi(w) = (\psi_1(w), \psi_2(w))$. It recognizes $S + T$ since the map $\pi : \mathfrak{A}_1 \times \mathfrak{A}_2 \to K$ defined by $\pi(x) = \pi_1(x) + \pi_2(x)$ is linear and such that $\pi(\psi(w)) = (S_1, w) + (S_2, w)$. By Lemma 1, ψ is completely reducible. Thus $\psi(K\langle A\rangle)$ is semisimple which implies that $S_1 + S_2$ is completely reducible. ■

4.2 The family of completely reducible sets

The syntactic representation (resp. algebra) of a set $X \subset A^*$ is the syntactic representation (resp. algebra) of its characteristic series.

A rational set is *completely reducible* if its characteristic series is completely reducible.

Example 5 The sets a^*, a^+ and 1 are completely reducible. Indeed, the syntactic algebras of a^* and 1 have dimension 1. Concerning a^+, the linear representation of Example 4 takes in the basis $1-2, 2$ the form $(\lambda', \mu', \gamma')$ with $\lambda' = [1\ 1]$, $\gamma' = [-1\ 1]^t$ and

$$\mu'(a) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Example 6 The sets $X = (ab)^*$ and $Y = (ab)^*a$ are completely reducible. Indeed, $\underline{X}$ is recognized by the linear representation (λ, μ, γ) with $\lambda = [1\ 0]$ $\gamma = [1\ 0]^t$ and

$$\mu(a) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \mu(b) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

Since there are no nontrivial invariant subspaces, the representation μ is completely reducible. The series $\underline{Y}$ is recognized by (λ, μ, γ') with $\gamma' = [0\ 1]^t$.

Example 7 The set $X = a$ is not completely reducible. Indeed, $\underline{X}$ is recognized by the linear representation (λ, μ, γ) with $\lambda = [1\ 0]$, $\gamma = [0\ 1]^t$ and

$$\mu(a) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

This representation is minimal. The subspace generated by $[0\ 1]$ is the only nontrivial invariant subspace. Thus X is not completely reducible.

Theorem 2 *The family of completely reducible sets is closed by residual, complement and reversal.*

Proposition 6 *The family of completely reducible sets is closed by reversal.*

Proof Let X be a completely reducible set. Let (λ, μ, γ) be a linear representation of $\underline{X}$. Let ν be the morphism from A^* into $K^{n\times n}$ defined by $\nu(w) = \mu(\tilde{w})^t$. Then $(\gamma^t, \nu, \lambda^t)$ is a linear representation of $\tilde{\underline{X}}$. Indeed

$$(\tilde{\underline{X}}, w) = (\underline{X}, \tilde{w}) = \lambda\mu(\tilde{w})\gamma = (\lambda\mu(\tilde{w})\gamma)^t = \gamma^t\nu(w)\lambda^t.$$

Moreover (λ, μ, γ) is minimal if and only if $(\gamma^t, \nu, \lambda^t)$ is minimal. ■

Proof of Theorem 2. Let $\mathcal{V}$ be the family of completely reducible sets.

For $X \in \mathcal{V}$ and $w \in A^*$, let $Y = w^{-1}X$. Set $S = \underline{X}$ and $T = \underline{Y}$. Then $T = S \cdot w$. The syntactic space V_T is the space generated by the $T \cdot u = S \cdot wu$. Since V_T is an invariant subspace of V_S, the invariant subspaces of V_T are invariant subspaces of V_S. Thus the syntactic representation of T is also completely reducible. Therefore $Y \in \mathcal{V}$. The proof that $Xw^{-1} \in A^*\mathcal{V}$ is similar, using Proposition 6.

Let $X \in \mathcal{V}$ and set $Y = A^* \setminus X$. Since A^* is completely reducible, by Proposition 15, the series $\underline{Y} = \underline{A^*} - \underline{X}$ is completely reducible. Thus $Y \in \mathcal{V}$.

■

The family of completely reducible sets is not closed intersection, as shown by Example 8. Since it is closed by complement is not closed by union either.

Example 8 Let $X = (ab)^*a$ and $Y = (ac)^*a$. The sets X and Y are completely reducible by Example 6. We have $X \cap Y = a$ which is not completely reducible by Example 7.

The following result shows an additional closure property.

Proposition 7 *For any rational set X, the sets X and $X \cap A^+$ are simultaneously completely reducible.*

Proof We may assume that $1 \in X$. Set $Y = X \cap A^+$. Since $\underline{Y} = \underline{X} - 1$, this results directly from Proposition 5. ■

4.3 Some properties of completely reducible sets

It follows from Proposition 4 that the syntactic algebra of a set X is a quotient of the algebra $K[M]$ where M is the syntactic monoid of X. As a consequence, we have the following statement, which gives a sufficient condition for complete reducibility.

Proposition 8 *If the algebra of the syntactic monoid of a set $X \subset A^*$ is semisimple, then X is completely reducible.*

A set $X \subset a^+$ is *periodic* of period $p \geq 1$ if for each $n \geq 1$, $a^n \in X$ if and only if $a^{n+p} \in X$.

The following result is a characterization of completely reducible sets on a one letter alphabet. We assume that the field K has characteristic 0. In the

proof, we use the following result: Let V be a finite dimensional vector space over K. An element x in $\mathrm{End}(V)$ generates a semisimple algebra if and only if the minimal polynomial of x has no factors of multiplicity > 1 over K (see [?] Chapter XVII, Exercise 10).

Theorem 3 *A rational set $X \subset a^*$ is completely reducible if and only if $X \cap A^+$ is periodic.*

Proof If $X \cap A^+$ is periodic of period p, the syntactic semigroup M of $X \cap A^+$ is isomorphic to $\mathbb{Z}/p\mathbb{Z}$. The algebra of M is semisimple since the algebra of the group $\mathbb{Z}/p\mathbb{Z}$ is semisimple. Thus $X \cap A^+$ is completely reducible. By Proposition 7 it implies that X is completely reducible.

Conversely, let X be a nonempty reducible subset of a^+. Set $S = \underline{X}$, $V = V_S$, $\psi = \psi_S$ and $\mathfrak{A} = \mathfrak{A}_S$. Let φ be the canonical morphism from a^* onto the the syntactic monoid M of X and let $m = \varphi(a)$. Let $i \geq 0$ and $p \geq 1$ be the index of M in such a way that

$$M = \{1, m, m^2, \ldots, m^{i-1}, m^i, \ldots, m^{i+p-1}\}$$

with $m^{i+p} = m^i$.

Set $x = \psi(a)$. Since $\mathfrak{A}$ is a quotient of $K[M]$, the minimal polynomial $f(t)$ of x divides $t^i(1 - t^p)$. Since $\mathfrak{A}$ is semisimple, the factor t has multiplicity at most 1 and thus $f(t)$ divides $t(1 - t^p)$. This shows that $x = x^{p+1}$.

This implies that, for any $n \geq 1$, $x^n = x^{n+p}$ and thus that

$$a^n \in X \Leftrightarrow (S, a^n) = 1 \Leftrightarrow (S \cdot a^n, 1) = 1 \Leftrightarrow (S \cdot a^{n+p}, 1) = 1 \Leftrightarrow a^{n+p} \in X.$$

Thus $X \cap A^+$ is periodic of period p. ■

Theorem 3 implies that the completely reducible subsets of a^* are of the form X or $X \cup 1$ for $X \subset a^+$ periodic.

We end the section with a necessary condition for complete reducibility. We say that a set X is *repeating* if for any $x \in X$ there exist words u, v such that $xuxv \in X$.

Proposition 9 *Any completely reducible rational set is repeating.*

Proof Arguing by contradiction, assume that X is not repeating. Let $x \in X$ be such that $xA^*xA^* \cap X = \emptyset$. Set $S = \underline{X}$ and $V = V_S$.

Let V' be the subspace of V generated by the series $S \cdot xu$ for $u \in A^*$. Note that for any element T of V', we have $T \cdot x = 0$. Indeed, if $T = \sum_{i=1}^n \alpha_i S \cdot xu_i$ for some $\alpha_i \in K$, we have $T \cdot x = \sum_{i=1}^n \alpha_i S \cdot xu_i x = 0$ since for any word u, we have $S \cdot xux = 0$.

We have $V' \neq 0$ because $(S \cdot x, 1) = 1$ and thus $S \cdot x$ is a nonzero element of V'. Since $S \cdot x \neq 0$, we have $S \notin V'$ and thus $V' \neq V$. By definition, V' is invariant. Assume that V' has an invariant complement V''. Then $S = S' + S''$ with $S' \in V'$ and $S'' \in V''$. Since $S' \cdot x = 0$, we have $S \cdot x = S'' \cdot x$. This

implies that $S'' \cdot x$ is in V'. Since V'' invariant, we have also $S'' \cdot x \in V''$ and thus $S'' \cdot x = 0$. This implies that $S \cdot x = 0$, a contradiction. Thus X is not completely reducible. ■

EXAMPLE 9 Let $X = ab^*$. For $w = a$, the set $X \cap wA^*wA^*$ is reduced to a and therefore X is not repeating. This shows that X is not completely reducible.

5 Birecurrent sets

In this section we introduce the class of birecurrent sets and we prove their complete reducibility. In the first part we state the main result (Theorem 4). In the second one we introduce the notion of accessible reversal of an automaton used in the proof of Theorem 4. In the last part, we give the proof of Theorem 4.

5.1 Main result

A nonempty set X is called *recurrent* if its minimal automaton is strongly connected. It is said to be *birecurrent* if X and its reverse $\tilde{X}$ are recurrent.

The submonoid generated by a prefix code is recurrent. Indeed, let X be prefix code. The submonoid generated by X is right unitary, which means by definition that for any words u, v if $u, uv \in X^*$, then $v \in X^*$. This implies that for any $x \in X^*$, one has $x^{-1}X^* = X^*$. Thus the minimal automaton of X^* is of the form $\mathcal{A} = (Q, i, i)$ with a set of terminal states reduced to the initial state. Since $\mathcal{A}$ is trim, this implies that $\mathcal{A}$ is strongly connected.

Thus the submonoid generated by a bifix code is birecurrent. The following example shows that other cases occur.

EXAMPLE 10 Let $X = \{a, ba\}$. The set X is a prefix code which is not bifix. The submonoid X^* is birecurrent. Indeed, the minimal automata of X^* and $\tilde{X}^*$ are represented in Figure 4. Both are strongly connected.

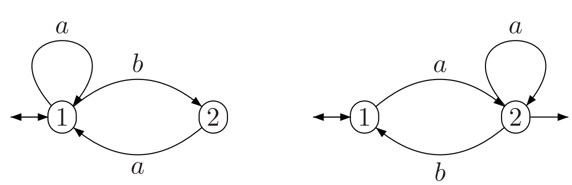

Figure 4: The minimal automata of X^* and $\tilde{X}^*$.

We will prove the following statement. We assume in this section that the field K is of characteristic 0.

Theorem 4 *A birecurrent set is completely reducible.*

Theorem 4 implies the following result, originally from [**?**], where the result is proved with a partial converse.

Corollary 1 *The submonoid generated by a rational bifix code is completely reducible.*

The proof of Theorem 4 uses Theorem 1. It is essentially the same as that given in [**?**] for Corollary 1.

5.2 Accessible reversal of an automaton

We begin this section with the following definition.

Let $\mathcal{A} = (Q, i, T)$ be a deterministic automaton. The *accessible reversal* of $\mathcal{A}$, denote by $\tilde{A}$, is the automaton obtained by successively

(i) reversing the edges of $\mathcal{A}$,

(ii) using the accessible subset construction to build an equivalent deterministic automaton using T as initial state and the subsets containing i as set of terminal states,

Thus the states of $\tilde{A} = (\tilde{Q}, T, J)$ are the nonempty sets

$$w^{-1}T = \{q \in Q \mid q \cdot w \in T\}.$$

This automaton recognizes $\tilde{X}$. Indeed, y is in $\tilde{X}$ if and only if $i \cdot \tilde{y} \in T$. And $i \cdot \tilde{y}$ is in T if and only if i is in $T \cdot y$ (for the transitions of $\tilde{A}$). Let $M = \varphi_{\mathcal{A}}(A^*)$ and $\tilde{M} = \varphi_{\tilde{\mathcal{A}}}(A^*)$ be the monoids of transitions of $\mathcal{A}$ and $\tilde{\mathcal{A}}$ respectively. There is an antiisomorphism $m \mapsto \tilde{m}$ from M onto $\tilde{M}$ such that the diagram below is commutative.

$$\begin{tikzcd}
A^* \arrow[r, "\sim"] \arrow[d, "\varphi_{\mathcal{A}}"] & A^* \arrow[d, "\varphi_{\tilde{\mathcal{A}}}"] \\
M \arrow[r, "\sim"] & \tilde{M}
\end{tikzcd}$$

In particular, for any word w, one has $m = \varphi_{\mathcal{A}}(w)$ if and only if $\tilde{m} = \varphi_{\tilde{\mathcal{A}}}(\tilde{w})$.

The action of M on the left on $\widetilde{Q}$ defined by $mU = V$ if $V = \{q \in Q \mid qm \in U\}$ is such that

$$mU = V \Leftrightarrow U\tilde{m} = V. \tag{2}$$

The following statement is well known (see [**?**] p. 48).

Proposition 10 *If $\mathcal{A}$ is a trim deterministic automaton recognizing X, then $\tilde{\mathcal{A}}$ is the minimal automaton of $\tilde{X}$.*

Proof Since $\mathcal{A}$ is trim, for any word w, one has $w^{-1}T \neq \emptyset$ if and only if $Xw^{-1} \neq \emptyset$. Moreover, for any $w, w' \in A^*$, one has

$$w^{-1}T = w'^{-1}T \Leftrightarrow Xw^{-1} = Xw'^{-1}$$

as one may easily verify. Since the nonempty sets Xw^{-1} are the reversals of the states of the minimal automaton $\tilde{X}$, the map $w^{-1}T \mapsto \tilde{w}^{-1}\tilde{X}$ is a bijection which identifies $\tilde{\mathcal{A}}$ with the minimal automaton of $\tilde{X}$. ∎

Thus, in particular, if $\mathcal{A}$ is the minimal automaton of X, then $\tilde{\mathcal{A}}$ is the minimal automaton of $\tilde{X}$.

Example 11 Let $\mathcal{A} = (Q, i, T)$ with $Q = \{1, 2, 3, 4\}$, $i = 1$ and $T = \{1, 2\}$ be the strongly connected automaton represented on the left in Figure 5. The

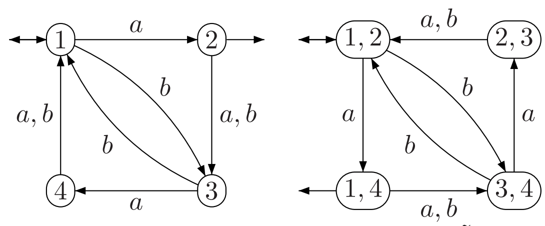

Figure 5: The automata $\mathcal{A}$ and $\tilde{\mathcal{A}}$.

accessible reversal $\tilde{\mathcal{A}}$ of $\mathcal{A}$ is represented in Figure 5 on the right. Since $\tilde{\mathcal{A}}$ is strongly connected, X is birecurrent. Note that X is not a submonoid since $a, abb \in X$ although $aabb \notin X$.

5.3 Proof of the main result

We begin with two preliminary statements.

Proposition 11 *Let $\mathcal{A} = (Q, i, T)$ be the minimal automaton of a set X. Set $S = \underline{X}$, $\tilde{S} = \underline{\tilde{X}}$ and $\varphi = \varphi_{\mathcal{A}}$. For any word $x \in A^*$, one has*

(i) *$i\varphi(x) = i$ if and only if $S \cdot x = S$,*

(ii) *$\varphi(x)T = T$ if and only if $\tilde{S} \cdot \tilde{x} = \tilde{S}$.*

Proof Assume that $i \cdot x = i$. Then, for any $u \in A^*$,

$$(S \cdot x, u) = 1 \Leftrightarrow xu \in X \Leftrightarrow i \cdot xu \in T \Leftrightarrow i \cdot u \in T \Leftrightarrow (S, u) = 1.$$

Thus $S \cdot x = S$. Conversely, if $S \cdot x = S$, then for any $u \in A^*$,

$$i \cdot xu \in T \Leftrightarrow (S, xu) = 1 \Leftrightarrow (S \cdot x, u) = 1 \Leftrightarrow (S, u) = 1 \Leftrightarrow i \cdot u \in T$$

which implies that $x^{-1}X = X$. In view of the definition of the minimal automaton, this shows that $i \cdot x = i$. Thus proves (i). The proof of (ii) is the same, using the fact that, by (2), one has $\varphi(x)T = T$ if and only if $T \cdot \tilde{x} = T$ in the automaton $\tilde{\mathcal{A}}$. ∎

Proposition 12 *Let $\mathcal{A} = (Q, i, T)$ be the minimal automaton of a birecurrent set X. Set $\varphi = \varphi_{\mathcal{A}}$ and $M = \varphi(A^*)$. The monoid M contains an idempotent e such that*

(i) *$ie = i$ and $eT = T$.*

(ii) *The set eMe is the union of a finite group G and of the element 0, provided $0 \in M$.*

Proof We assume that M contains a zero. The other case is similar. By Proposition 2, the monoid M has a unique 0-minimal two-sided ideal D which is a regular $\mathcal{D}$-class. Let w be a word such that $\varphi(w)$ belongs to D. Since $\mathcal{A}$ is strongly connected there is a word u such that $i \in Q \cdot wu$. Set $w' = wu$. Then $\varphi(w')$ is in D. Next, since $\tilde{A}$ is strongly connected, there is a v such that $T \cdot \tilde{w}'v = T$ and thus $\varphi(\tilde{v}w')T = T$. Then $\varphi(\tilde{v}w')$ is in D. Since $\varphi(\tilde{v}w')T = T$, we cannot have $\varphi(\tilde{v}w')^2 = 0$ and thus there is a power x of $\tilde{v}w'$ such that $e = \varphi(x)$ is an idempotent. Since $i \in Q \cdot w'$, i is in the image of e and thus we have $ie = i$ since e is idempotent of D. We have also $eT = T$ Moreover the non zero elements of the set eMe form the group of the $\mathcal{D}$-class D.

■

The group G defined above is called the Suschkevitch group of the monoid M. It is the group of the 0-minimal ideal of M.

Example 12 Consider again the birecurrent set of Example 11. The minimal ideal of M is represented in Figure 6. The idempotent $e = \varphi_{\mathcal{A}}(b^2)$ is such that

	$1,3$	$2,4$
$1,2/3,4$	$^*\ b$	$^*\ ba$
$1,4/2,3$	$^*\ ab$	$^*\ aba$

Figure 6: The 0-minimal ideal of M.

$1e = 1$ and $eT = T$. The set eMe is the group $\mathbb{Z}/2\mathbb{Z}$.

Proof of Theorem 4. Let X be a birecurrent set and let $S = \underline{X}$. Let $\mathcal{A} = (Q, i, T)$ be the minimal automaton of X.

Set $\varphi = \varphi_{\mathcal{A}}$ and $\psi = \psi_S$. By Proposition 12, there exists a word $x \in A^*$ such that $i\varphi(x) = i$ and $\varphi(x)T = T$ and such that $\varphi(xA^*x)$ is the union of 0 (if $0 \in \varphi(A^*)$) and of a finite group.

Set $M = \psi(A^*)$ and $e = \psi(x)$. By Proposition 4, e is an idempotent of M such that eMe is the union of 0 (if $0 \in M$) and of a finite group. Moreover, by Proposition 11 and its dual, we have $(S, u) = (S, ux) = (S, xu)$ for any $u \in A^*$.

Set $V = V_S$. Taking a basis of V, we may consider M as a monoid of $n \times n$-matrices and V as the space of row n-vectors. Let λ be the row n-vector

representing S and let γ be the column n-vector such that $(S,w) = \lambda\psi(w)\gamma$ for all $w \in A^*$.

We verify that e satisfies the hypotheses of Theorem 1. Condition (i) is satisfied by Mashke's teorem. Next, since $i\varphi(x) = i$, we have $\lambda e = \lambda$ by Proposition 11. Since V is generated by the vectors λm for $m \in M$, it is generated by the set $\lambda e M$. Thus condition (ii) is satisfied.

Let W be the space of column n-vectors. Symmetrically to the fact that V is generated by the elements of the set λm, for $m \in M$, the space W is generated by the elements of the set $m\gamma$ for $m \in M$. By assertion (ii) of Proposition 11, since $\varphi(x)T = T$, we have $e\gamma = \gamma$. Thus W is generated by the elements of the set $Me\gamma$. This shows that condition (iii) is satisfied.

By Theorem 1, the monoid M is completely reducible and thus the proof is complete. ∎

Note that for any birecurrent set X, by Theorem 1, the irreducible components of the syntactic representation of X are in bijection with the irreducible components of the permutation representation of the group. We illustrate this in the following example.

Example 13 Consider again the birecurrent set of Example 11. The syntactic representation of X is obtained from the linear representation associated with the automaton $\mathcal{A}$ after taking the quotient of the space K^Q by the subspace generated by $1+3-2-4$. Thus, in the basis $1,2,3$, we have

$$\psi(a) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix}, \quad \psi(b) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

The subspace generated by the vector $1+3$ is invariant. It has an invariant complement formed of the vectors with zero sum of coefficients. In the basis $1+3, 1-3, 2-4$, the matrices $\psi(a), \psi(b)$ take the following form.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

Thus the syntactic representation of X is the sum of of two irreducible representations of dimensions 1 and 2.

6 Cyclic sets

In the first part of this section, we recall the definition of a cyclic set which was introduced in [?]. In the second part, we give a new proof of their complete reducibility.

6.1 Cyclic and strongly cyclic sets

A subset X of a monoid M is *cyclic* if it satisfies the two following conditions.

(i) For any $u, v \in M$, one has $uv \in X$ if and only if $vu \in X$.

(ii) For any $w \in M$ and any integer $n \geq 1$, one has $w^n \in X$ if and only if $w \in X$.

If φ is a morphism from a monoid M onto a monoid N, for any subset X of N, the set $\varphi^{-1}(X)$ is cyclic if and only if X is cyclic.

Example 14 The cyclic subsets of a^* are the sets $\emptyset$, 1, a^+ and a^*.

A rational set of words X is *strongly cyclic* if there is a morphism φ from A^* into a finite monoid M which has a zero such that $X = \{x \in M \mid 0 \notin \varphi(x^*)\}$. Let $\mathcal{A}$ be a deterministic automaton with a set Q of states. The set of cyclically nonzero words defined by $\mathcal{A}$ is the set

$$X = \{x \in A^* \mid Q \cdot x^n \neq \emptyset \text{ for all } n \geq 0\}. \tag{3}$$

Note that since Q is finite, for any $x \in X$ there is a $q \in Q$ such that $q \cdot x^n \neq 0$ for all $n \geq 0$.

Proposition 13 *A set of words X is strongly cyclic if and only if it is the set of cyclically nonzero words defined by a deterministic automaton.*

Proof The condition is necessary. Indeed, let $\varphi : A^* \to M$ be a morphism into a finite monoid M which has a zero such that $X = \{x \in M \mid 0 \notin \varphi(x^*)\}$. Let $\mathcal{A}$ be the automaton with $M \setminus 0$ as set of states and with transitions defined by $m \cdot a = m\varphi(a)$ if $m\varphi(a) \neq 0$. For any $x \in X$, one has $1 \cdot x^n \neq \emptyset$ for all $n \geq 0$. Thus x satisfies condition (3). Conversely, if $m \cdot x^n \neq \emptyset$ for some $m \in M \setminus 0$ and for all $n \geq 0$, then $0 \notin \varphi(x^*)$.

The condition is also sufficient. Indeed, assume that X is the set of cyclically nonzero words defined by the deterministic automaton $\mathcal{A}$. Let M be the transition monoid of $\mathcal{A}$ and let φ be the canonical morphism from A^* onto M. For any $x \in X$, one has $\varphi(x^n) \neq 0$ and thus $0 \notin \varphi(x^*)$. Conversely, if $0 \notin \varphi(x^*)$, let k be an integer such that $\varphi(x^k)$ is idempotent. Since $\varphi(x^k) \neq 0$, there is a state q such that $q \cdot x^k \neq \emptyset$. Then $q \cdot x^{kn} \neq \emptyset$ for any $n \geq 0$ and consequently $q \cdot x^n \neq \emptyset$ for any $n \geq 0$. Thus X is strongly cyclic. ∎

For a sequence $X_1, \ldots, X_n$ such that $X_1 \supset X_2 \supset \ldots \supset X_n$, the *chain of differences* of the sequence is the set

$$X = (X_1 - X_2) + (X_3 - X_4) + \ldots. \tag{4}$$

The integer n is called the length of the chain. According to the parity of n the last term of the chain is $(X_{n-1} - X_n)$ or (X_n). Note that one can also write (4)

as $X = X_1 - Y$ with $Y = (X_2 - X_3) + (X_4 - X_5) + \dots$ a chain of differences of length $n-1$ such that $Y \subset X$.

The following result is from [?] (see the proof of Theorem 10). It shows in particular that any cyclic rational language is a boolean combination of strongly cyclic rational languages.

Proposition 14 *Any cyclic rational language X is a chain of differences of strongly cyclic rational languages.*

Example 15 Consider the automaton $\mathcal{A}$, called the *even automaton*, represented in Figure 7 on the left. The automaton on the right will be used below. Let X be the set of cyclically nonzero words for this automaton. We have

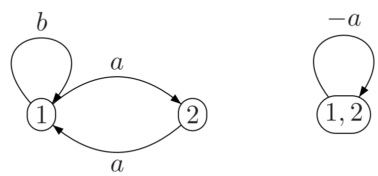

Figure 7: The even automaton

$$X = a^* \cup (a^2)^* b\{aa, b\}^* \cup a(a^2)^* b\{aa, b\}^* a.$$

The set X is the union of two cyclic sets a^* and $(a^2)^* b\{aa, b\}^* \cup a(a^2)^* b\{aa, b\}^*$. The first one is strongly cyclic but the second is not.

6.2 Cyclic sets are completely reducible

The following result is from [?] (Corollary 12.2.2 [?]).

Theorem 5 *A rational cyclic set is completely reducible.*

A series S is a *trace series* if there exists a linear representation μ of A^* such that for any $w \in A^*$

$$(S, w) = \mathrm{Tr}(\mu w).$$

The following is from [?] (it is Lemma 12.2.3 in [?]).

Proposition 15 *The syntactic algebra of a linear combination of trace series is semisimple.*

Let $\mathcal{A}$ be a finite deterministic automaton with set of states $Q = \{1, 2, \dots, n\}$ on the alphabet A. Following [?], for $1 \le k \le n$, the *external power* of order k of $\mathcal{A}$ is the weighted automaton $\mathcal{A}_k$ defined as follows. Its set of states is the set Q_k of sequences of integers $(i_1, i_2, \dots, i_k)$ such that $1 \le i_1 < i_2 < \dots < i_k \le n$. The edges are labeled in $A \cup -A$. There is a transition by εa from $(i_1, i_2, \dots, i_k)$ to $(j_1, j_2, \dots, j_k)$ if and only if $(j_1, j_2, \dots, j_k)$ is obtained from $(i_1 \cdot a, i_2 \cdot a, \dots, i_k \cdot a)$ by a permutation of signature ε.

Example 16 Let $\mathcal{A}$ be the even automaton of Example 15. The external power $\mathcal{A}_2$ is represented in Figure 7 on the right.

The following combinatorial lemma on permutations is Lemma 6.4.9 in [**?**].

Lemma 2 *Let π be a permutation of a finite set P and let $\mathcal{R} = \{R \subset P \mid R \neq \emptyset, \pi(R) = R\}$. Then*

$$\sum_{R \in \mathcal{R}} (-1)^{\mathrm{Card}(R)+1} \varepsilon(\pi, R) = 1$$

where $\varepsilon(\pi, R)$ denotes the signature of the restriction of π to the set R.

We use Lemma 2 to prove the following result.

Proposition 16 *If X is a strongly cyclic rational set, the series $\underline{X}$ is a linear combination of trace series.*

Proof Let $\mathcal{A}$ be a deterministic automaton on the set $Q = \{1, 2, \ldots n\}$ such that X is the set of cyclically nonzero words defined by $\mathcal{A}$. Denote by $\mathcal{A}_i$ for $1 \leq i \leq n$ the external power of $\mathcal{A}$ of order k. We denote by $\mathrm{Tr}_i(w)$ the trace of a word w with respect to the automaton $\mathcal{A}_i$. We have

$$\mathrm{Tr}_i(w) = \sum_{q \in Q_{i,w}} \varepsilon_{q,w} \tag{5}$$

where $Q_{i,w}$ is the set of $q \in Q_i$ such that $q \cdot w$ differs from q by a permutation of signature $\varepsilon_{q,w}$.

We claim that for each $x \in A^*$

$$(\underline{X}, x) = \sum_{i=1}^{n} (-1)^{i+1} \, \mathrm{Tr}_i(x). \tag{6}$$

This will imply the result by Proposition 15.

To prove (6), assume first that $x \in X$. Let $P \subset Q$ be the largest set such that x defines a permutation π of P. Since x is cyclically nonzero, P is not empty. For each $i = 1, \ldots n$, $\mathrm{Tr}_i(x) = \sum_{q \in Q_{i,x}} \varepsilon_{q,x}$ by Equation (5). But the set $Q_{i,x}$ is the set of sequences $q = (q_1, \ldots, q_i)$ with $q_1 < \ldots < q_i$ such that the set $R = \{q_1, \ldots, q_i\}$ satisfies $\pi(R) = R$. These sequences are thus in bijection with the sets R in $\mathcal{R} = \{R \subset P \mid R \neq \emptyset, \pi(R) = R\}$. Thus

$$\sum_{i=1}^{n} (-1)^{i+1} \, \mathrm{Tr}_i(x) = \sum_{R \in \mathcal{R}} (-1)^{\mathrm{Card}(R)+1} \epsilon(\pi, R)$$

By Lemma 2 the value of the right hand side is 1. Thus we have proved (6) for $x \in X$.

Next if $x \notin X$, then $\mathrm{Tr}_i(x) = 0$ for all $i = 1, \ldots n$. Indeed, if $\mathrm{Tr}_i(x) \neq 0$, there is a sequence $q_1, \ldots, q_i$ such that $q_1 \cdot x = q_1, \ldots, q_i \cdot x = q_i$ and thus $x \in X$. Thus the right handside of (6) is zero. This proves (6) for $x \notin X$. ■

Proof of Theorem 5. By Proposition 14, any cyclic rational set is a chain of differences of strongly cyclic sets. We prove by induction on the length n of the chain that for any cyclic rational set X, the series $\underline{X}$ is a linear combination of trace series. By Proposition 15 it implies the conclusion.

It is true when $n = 0$ since then X is empty.

Assume now that $n \geq 1$. Then $X = Y - Z$ where Y is strongly cyclic and $Z \subset Y$ is a chain of differences of length $n-1$ of strongly cyclic sets. by Proposition 16 $\underline{Y}$ is a linear combination of trace series. By induction hypothesis, $\underline{Z}$ is a linear combination of trace series. Since $\underline{X} = \underline{Y} - \underline{Z}$, the same conclusion holds for $\underline{X}$. ■

Example 17 Consider again the even automaton $\mathcal{A}$ represented in Figure 7 on the left. Let X be the set of cyclically nonzero words for $\mathcal{A}$.

The minimal automaton of X is represented in Figure 8

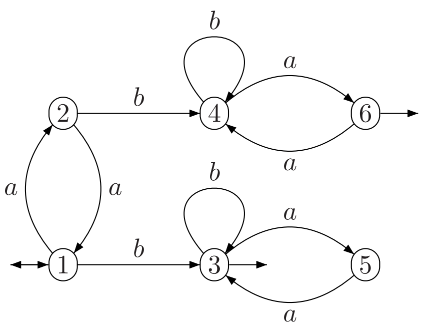

Figure 8: The minimal automaton of X.

To obtain the syntactic representation of $\underline{X}$ we write the linear representation associated with the automaton $\mathcal{A}$ in the basis formed of the row vectors

$$1 - 3 - 6, \quad 2 - 4 - 5, \quad 3, \quad 5, \quad 4, \quad 6$$

$$\psi(a) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad \psi(b) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The initial and terminal vectors in this basis are

$$\lambda = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}, \quad \gamma = \begin{bmatrix} -1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}^t$$

In this way, the representation is a direct sum of three representations of degrees $2, 2, 2$. The first one is equivalent to a representation of degree 1.

Thus the syntactic representation is the direct sum of three representations of dimensions $1, 2, 2$. The first one is the linear representation associated with $\mathcal{A}_2$. The two other ones are equal to the linear representation associated with $\mathcal{A}$ in such a way that the pair recognizes the trace of the associated matrices.